3D Implementation of Continuous Interior Penalty FEM for Incompressible Navier-Stokes Equations (with Free Surface)

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Motivation

EPFL is the scientific partner of Team Alinghi for the America's Cup 2004 as well as 2007. The prediction of the boat's resistance, including not only viscous but also wake effects, is a part of this collaboration.
The following approaches for treating the free boundary have been studied:

- **ALE** (arbitrary Lagrangian-Eulerian) with moving mesh whose boundary follows the air-water interface. Problem: Breaking waves

- **Volume of Fluid** method. Problem: interface smears out over time
- **Level Set** method: investigated by N. Parolini (EPFL – CMCS, see e. g. [3]) in 2D, our first steps towards 3D presented here

Model Equations

Incompressible Navier-Stokes equations with variable density ρ and viscosity μ

$$\rho(\underline{x},t) = \begin{cases} \rho_- & \text{if } \underline{x} \in \Omega_-(t) \\ \rho_+ & \text{if } \underline{x} \in \Omega_+(t) \end{cases}$$

idem for μ.
consider a domain $\Omega \subset I\!R^3$

$$\begin{aligned} \rho\partial_t\underline{u} + \rho(\underline{u}\cdot\underline{\nabla})\underline{u} - \underline{\nabla}\cdot(2\mu\underline{\underline{\varepsilon}}(\underline{u})) + \underline{\nabla}p &= \rho\underline{g} \\ \underline{\nabla}\cdot\underline{u} &= 0 \\ \partial_t\phi + \underline{u}\cdot\underline{\nabla}\phi &= 0 \end{aligned}$$

in Ω_- and in Ω_+, $\bar{\Omega} = \bar{\Omega}_-(t) \cup \bar{\Omega}_+(t)$.
The level set function ϕ carries the information about the position of the interface Γ:

$$\begin{aligned} \Gamma(t) &= \{\underline{x} : \phi(\underline{x},t) = 0\} \\ \Omega_\pm(t) &= \{\underline{x} : \pm\phi(\underline{x},t) > 0\} \end{aligned}$$

interface conditions at interface Γ:

$$[\![\underline{u}]\!]_\Gamma = \underline{0}, \qquad [\![2\mu\underline{\underline{\varepsilon}}(\underline{u}) - p\underline{\underline{I}}]\!]\cdot\underline{n}_\Gamma = \kappa\sigma\underline{n}_\Gamma$$

κ: interface curvature
σ: surface tension
Boundary and initial conditions:

$$\begin{aligned} \underline{u} &= \underline{0} \quad \text{on } \partial\Omega\times(0,T) \\ \underline{u} &= \underline{u}_0 \quad \text{in } \Omega \text{ at } t=0 \\ \phi &= \phi_0 \quad \text{in } \Omega \text{ at } t=0 \end{aligned}$$

Variational Formulation

Find $\underline{u}$, p and ϕ such that

$$\begin{aligned} (\rho\partial_t\underline{u},\underline{v}) + a(\underline{u},\phi;\underline{u},\underline{v}) + b(p,\underline{v}) &= (\rho\underline{g},\underline{v}) \\ b(q,\underline{u}) &= 0 \\ (\partial_t\phi,\psi) + c(\underline{u};\phi,\psi) &= 0 \end{aligned}$$

$\forall\underline{v}$, q, and ψ
with

$$\begin{aligned} a(\underline{\beta},\phi;\underline{u},\underline{v}) &:= (\rho(\phi)(\underline{\beta}\cdot\underline{\nabla})\underline{u},\underline{v}) \\ &\quad +2(\mu(\phi)\underline{\underline{\varepsilon}}(\underline{u}),\underline{\underline{\varepsilon}}(\underline{v})) \\ b(p,\underline{v}) &:= -(p,\underline{\nabla}\cdot\underline{v}) \\ c(\underline{\beta};\phi,\psi) &:= (\underline{\beta}\cdot\underline{\nabla}\phi,\psi) \end{aligned}$$

CIP Stabilization

Idea: penalize jumps of gradients of continuous FE functions [1]

$$\begin{aligned} j_{\underline{u}}(\underline{\beta};\underline{u},\underline{v}) &:= \sum_{K\in\mathcal{T}_h}\gamma_{\underline{\beta}}\frac{\rho h_K^2}{\|\underline{\beta}\|_{0,\infty,K}} \int_{\partial K}(\underline{\beta}\cdot[\![\underline{\nabla}\,\underline{u}]\!])\cdot(\underline{\beta}\cdot[\![\underline{\nabla}\,\underline{v}]\!])\mathrm{d}s \\ &\quad + \sum_{K\in\mathcal{T}_h}\gamma_{\mathrm{div}}\rho h_K^2\|\underline{\beta}\|_{0,\infty,K} \int_{\partial K}[\![\underline{\nabla}\cdot\underline{u}]\!][\![\underline{\nabla}\cdot\underline{v}]\!]\mathrm{d}s \\ j_p(\underline{\beta};p,q) &:= \sum_{K\in\mathcal{T}_h}\gamma_p\frac{h_K^3}{\max\{\rho h_K\|\underline{\beta}\|_{0,\infty,K},\mu\}} \int_{\partial K}[\![\underline{\nabla}p]\!]\cdot[\![\underline{\nabla}q]\!]\mathrm{d}s \\ j_\phi(\underline{\beta};\phi,\psi) &:= \sum_{K\in\mathcal{T}_h}\gamma_\phi\frac{h_K^2\|\underline{\beta}\cdot\underline{n}\|_{0,\infty,K}^2}{\|\underline{\beta}\|_{0,\infty,K}} \int_{\partial K}[\![\underline{\nabla}\phi]\!]\cdot[\![\underline{\nabla}\psi]\!]\mathrm{d}s \end{aligned}$$

$$\begin{aligned} \mathcal{A}(\underline{\beta};(\underline{u},p),(\underline{v},q)) &:= a(\underline{\beta};\underline{u},\underline{v}) + b(p,\underline{v}) \\ &\quad + j_{\underline{u}}(\underline{\beta};\underline{u},\underline{v}) \\ &\quad - b(q,\underline{u}) + j_p(\underline{\beta};p,q) \\ \mathcal{C}(\underline{\beta};\phi,\psi) &:= c(\underline{\beta};\phi,\psi) + j_\phi(\underline{\beta},\phi,\psi) \end{aligned}$$

Allows for

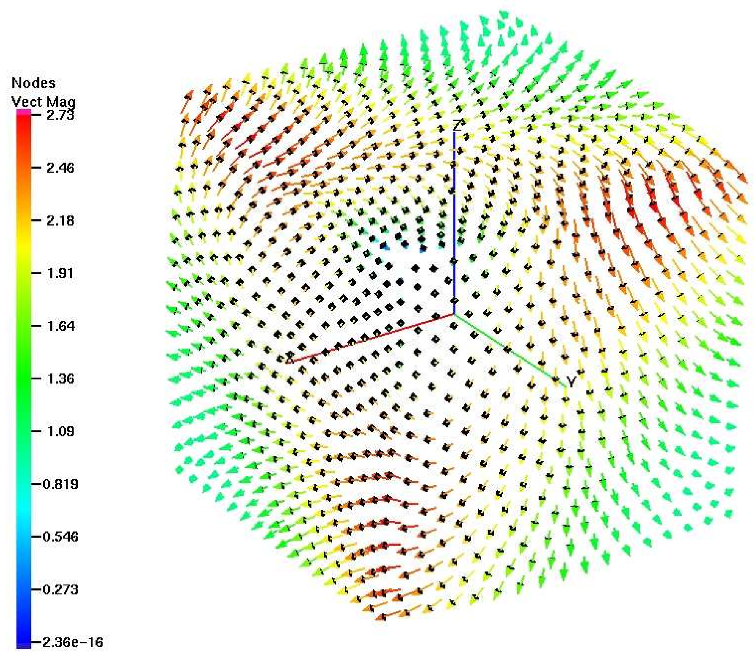
- satisfaction of inf-sup condition for equal order spaces
- stabilization of the incompressibility constraint
- stabilization of dominant convection

Discretization

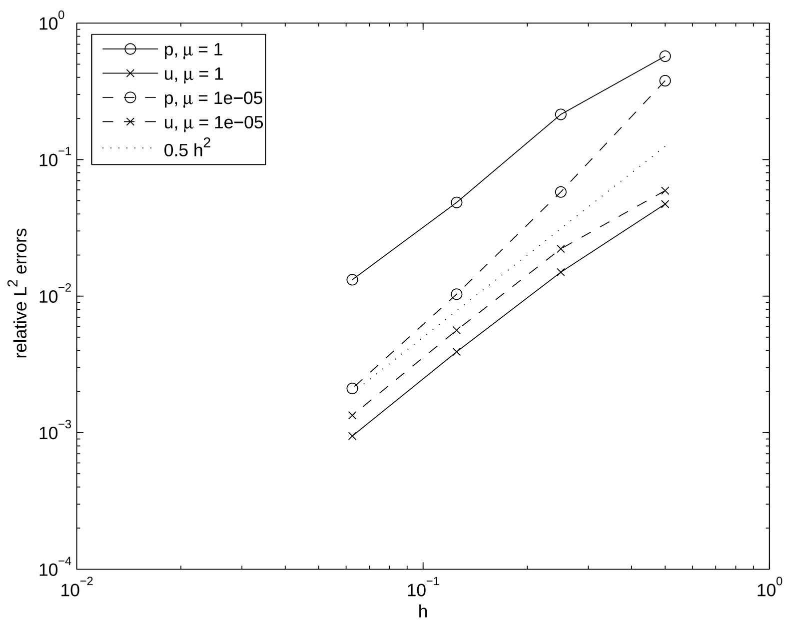
- $\mathcal{P}_1$ finite elements in space for u, p and ϕ
- BDF2 in time, $\underline{\beta}$ extrapolated
- first order decoupling of Navier-Stokes ($\mathcal{A}$) and interface advection ($\mathcal{C}$)

Numerical Results

Test for one fluid with smooth exact 3D solution by Ethier/Steinman [2]
$\Omega = [-1,1]^3$, $\rho = 1$, $\mu = 1$ and $\mu = 10^{-5}$
$\nu = \mu/\rho$, $T = 0.1$, $\Delta t = 0.025$
$\gamma_{\underline{\beta}} = 0.02$, $\gamma_{div} = \gamma_p = 0.2$



Velocity field



Second order convergence of velocity and pressure in L^2 for low and high Reynolds numbers (laminar)

References

[1] Erik Burman, Miguel A. Fernández, and Peter Hansbo. Edge stabilization for the incompressible Navier-Stokes equations: a continuous interior penalty finite element method. http://iacs.epfl.ch/~burman/CIP_NS.pdf, 2004.

[2] C. Ross Ethier and D. A. Steinman. Exact fully 3D Navier-Stokes solutions for benchmarking. *Int. J. Numer. Meth. Fluids*, 19:369–375, 1994.

[3] Nicola Parolini. *Computational Fluid Dynamics for Naval Engineering Problems*. PhD thesis, Ecole Polytechnique Fédérale de Lausanne, 2004.

First numerical tests with free surface

Rising bubble
density ratio: 2
viscosity ratio: 1
no surface tension
Observations:
- reflux around bubble
- bubble takes characteristic form
- artifacts due to very coarse mesh
Joint work with D. A. di Pietro, Università degli Studi di Bergamo